One-dimensional asymmetric diffusion model without exclusion

Tomohiro Sasamoto and Miki Wadati
Department of Physics, Graduate School of Science, University of Tokyo, Hongo 7-3-1, Bunkyo-ku, Tokyo 113-0033, Japan
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A one-dimensional asymmetric diffusion model on a periodic chain is proposed. The model is defined in terms of the master equation. In contrast with the asymmetric simple exclusion process, particles are not subject to the exclusion interaction: each lattice site can accommodate more than one particle. The model is solved by the Bethe ansatz method and the resulting Bethe equation is analyzed in the thermodynamic limit. The finite size correction of the energy gap is calculated to be $O(L^{-3/2})$, where L denotes the length of the chain. [S1063-651X(98)12109-8]

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I. INTRODUCTION

In a variety of fields of physics, chemistry, and biology, there exist a lot of phenomena that can be well described by stochastic models of many particles. It is interesting that highly nontrivial behaviors of systems at far-from equilibrium can sometimes be explained by stochastic models defined by rather simple rules. For instance, the one-dimensional asymmetric simple exclusion process (ASEP) is known to be relevant to the problems such as interface growth and traffic flow [1–3]. The ASEP is a lattice model of particles that hop to nearest-neighboring sites stochastically. Each particle moves to the right (left) nearest neighboring site with a probability $D_R dt$ ($D_L dt$) in an infinitesimal interval dt. Without loss of generality we assume $0 \leqslant D_L \leqslant D_R$ hereafter. In addition, particles are subject to hard-core exclusion: each site is either occupied only by one particle or empty. The ASEP may seem a too simplified model since the interaction among particles is only through the hard-core exclusion. It shows, however, rich nonequilibrium behaviors and has been intensively studied by many researchers.

An important feature of one-dimensional stochastic models is that we can sometimes obtain exact solutions by analytic methods, for instance, the Bethe ansatz and free fermion techniques [4,5]. To apply such methods, it is convenient to formulate the problems by the master equation in the form of the imaginary-time Schrödinger equation,

$$\frac{d}{dt}P = -HP. \tag{1}$$

Here P and H symbolically denote the probability distribution of the system and the transition rate matrix, respectively. Although H is in general non-Hermitian, we call H the Hamiltonian. The ASEP is known to be exactly solvable since the Hamiltonian of the ASEP is connected to the Heisenberg Hamiltonian for magnets through a similarity transformation. To be more precise, the ASEP is a special case of the asymmetric XXZ chain, which is a non-Hermitian generalization of the well-known XXZ chain [6–9]. Besides, the asymmetric XXZ chain is related to the asymmetric six-vertex model, which is interpreted as the six-vertex model in electric fields [10–12]. The Boltzmann weights of the asymmetric six-vertex model are connected to those of the usual six-vertex model through a gauge transformation. Hence the Boltzmann weights satisfy the Yang-Baxter equation and the model is easily shown to be integrable. Interestingly, the physical properties of the asymmetric six-vertex model and the asymmetric XXZ chain are much richer than the original six-vertex model and the XXZ chain. They are still under extensive investigations [9,13–15].

The Bethe equation of the asymmetric XXZ chain can be solved exactly in the thermodynamic limit [16,17]. Moreover, the analysis of the Bethe equation enables us to obtain the finite size corrections of the low-lying energies of the asymmetric XXZ chain [6–9]. In particular, for the ASEP, the energy gap was shown to scale as $O(L^{-3/2})$ with L being the length of the chain. The exponent 3/2 is the same as the dynamic critical exponent of the Kardar-Parisi-Zhang (KPZ) equation in one dimension [18], which is a nonlinear stochastic equation for the height of growing surfaces. This is plausible since the ASEP is believed to be a discretized version of the noisy Burgers equation [6]. The noisy Burgers equation and the KPZ equation are related to each other by the change of the variable from ''height'' to ''slope'' of growing surfaces. Thus the ASEP is considered to belong to the KPZ universality class.

The above-mentioned scaling is considered as a manifestation of the anisotropic critical phenomena [19,20]. They often appear in non-Hermitian problems but have not been well understood compared with the isotropic one. For instance, while the hard-core exclusion interaction is regarded as the origin of the interesting behaviors of the ASEP, the problem of, i.e., what kind of interactions among particles bring the process into the KPZ universality class, has been less addressed. Hence more explicit model studies seem important to understand what the universality means for stochastic models. In Refs. [21,22], an asymmetric diffusion model without exclusion was shown to be integrable and to have the same R matrix as that of the ASEP. The Hamiltonian of the model is defined in terms of the so-called q-boson operators. The q-boson model is expected to belong to the KPZ universality class since it shares the same R matrix with the ASEP. However, the Bethe equation for the model is somewhat different from that for the ASEP and needs some modifications to analyze.

The main purpose of this paper is to show that we can define an asymmetric diffusion model without exclusion, for which the Bethe ansatz method is applicable and the resulting Bethe equation is solvable parallel to the ASEP case. Basically, the model is similar to the ASEP: each particle is an asymmetric random walker. However, more particles than one can be on the same site and even hop simultaneously to a nearest-neighboring site. In spite of these differences, the model and the ASEP share quite similar Bethe equations and hence similar physical properties. The finite size correction of the energy gap is calculated to be $O(L^{-3/2})$, which is the same as that for the ASEP. This suggests that the model presented in this paper belongs to the KPZ universality class.

The plan of this paper is as follows. In the next section, an asymmetric diffusion model without exclusion is defined. The model is shown to be solvable by the Bethe ansatz in Sec. III and the resulting Bethe equation is analyzed in Sec. IV. In Sec. V, the finite size correction of the energy gap is calculated. The last section is devoted to the concluding remarks.

II. ASYMMETRIC DIFFUSION MODEL WITHOUT EXCLUSION

Consider a one-dimensional lattice with periodic boundary condition. We introduce a model with particle number conservation. With the particle number N fixed, let $P_N(x_1,\ldots,x_N;t)$ denote the probability that the particles are located at lattice sites $x_1,\ldots,x_N$ at time t. Since we do not distinguish one particle from another, we assume $x_1 \leqslant x_2 \leqslant \cdots \leqslant x_N$. We define the process in terms of the master equation for $P_N(x_1,\ldots,x_N;t)$ in the following. For comparison, the definition of the ASEP will also be given.

Before giving the definition for general N, we proceed with the $N=1$ and $N=2$ cases. For one particle, the process is nothing but the asymmetric random walk in a continuous time. Putting $\gamma=D_L/D_R$ $(0\leqslant\gamma\leqslant1)$ and rescaling time, the master equation reads

$$\frac{d}{dt}P_1(x;t)=P_1(x-1;t)+\gamma P_1(x+1;t)-(1+\gamma)P_1(x;t), \tag{2}$$

which is common to the model and the ASEP. Symbolically, we represent the process (2) as

$$10\overset{1}{\rightarrow}01, \quad 01\overset{\gamma}{\rightarrow}10, \tag{3}$$

where ''0'' and ''1'' indicate the particle numbers on a site.

Next we consider the two-particle case. We assume that each particle performs the asymmetric random walk if the distance between the two particles is sufficiently large. Hence the master equation for $P_2(x_1,x_2;t)$ is

$$\begin{aligned}\frac{d}{dt}P_2(x_1,x_2;t)=&P_2(x_1-1,x_2;t)+\gamma P_2(x_1+1,x_2;t)\\&+P_2(x_1,x_2-1;t)+\gamma P_2(x_1,x_2+1;t)\\&-2(1+\gamma)P_2(x_1,x_2;t).\end{aligned} \tag{4}$$

TABLE I. Particle hopping rates for the MADM (multiparticle-hopping asymmetric diffusion model) for the two-particle case. The numbers ''0,'' ''1,'' and ''2'' in the leftmost boxes indicate the particle numbers on a site. For comparison, the rates for noninteracting particles are also shown.

	rate (MADM)	rate (noninteracting)
20→02	1/[2]	0
02→20	$\gamma^2/[2]$	0
20→11	1	2
02→11	γ	2 γ
11→02	1	1
11→20	γ	γ

For the ASEP, the hard-core exclusion restricts an allowed region (the physical region) of the coordinates x_1,x_2 to $x_2-x_1\geqslant1$. The above master equation (4) applies only for $x_2-x_1>1$. When $x_2-x_1=1$, we have to employ a slightly different equation due to the exclusion. The master equation for this case is given by

$$\begin{aligned}\frac{d}{dt}P_2(x,x+1;t)=&P_2(x-1,x+1;t)+\gamma P_2(x,x+2;t)\\&-(1+\gamma)P_2(x,x+1;t).\end{aligned} \tag{5}$$

We notice that the master equation (5) for the boundary of the physical region is equivalent to putting the condition

$$(1+\gamma)P_2(x,x+1;t)=P_2(x,x;t)+\gamma P_2(x+1,x+1;t) \tag{6}$$

in Eq. (4). It is only a matter of convenience whether we define the ASEP for the $N=2$ case by Eq. (4) for $x_2-x_1>1$ and Eq. (5) or by Eq. (4) for $x_2-x_1\geqslant1$ and Eq. (6).

Now, for $N=2$, we define the model to be considered in this paper. First, the physical region of the coordinates x_1,x_2 is $x_2\geqslant x_1$. When $x_1=x_2$, two particles are on the same site. The particles do not have hard-core exclusion interaction. If we put $x_1=x_2$ in the master equation (4), there appear the functions $P_2(x,x-1;t)$ and $P_2(x+1,x;t)$, of which the variables are out of the physical region. To avoid such inconsistencies, we impose the following condition:

$$(1+\gamma)P_2(x,x-1;t)=P_2(x-1,x-1;t)+\gamma P_2(x,x;t). \tag{7}$$

Then the master equation for $P_2(x,x;t)$ is shown to be

$$\begin{aligned}\frac{d}{dt}P_2(x,x;t)=&P_2(x-1,x;t)+\frac{1}{[2]}P_2(x-1,x-1;t)\\&+\gamma P_2(x,x+1;t)+\frac{\gamma^2}{[2]}P_2(x+1,x+1;t)\\&-\left(1+\frac{1}{[2]}+\gamma+\frac{\gamma^2}{[2]}\right)P_2(x,x;t),\end{aligned} \tag{8}$$

where $[2]=1+\gamma$. Similar to the ASEP case, it is equivalent to define the model either by Eq. (4) for $x_2>x_1$ and Eq. (8) or by Eq. (4) for $x_2\geqslant x_1$ and Eq. (7). To make the defined process clear, we list the particle hopping rates other than Eq. (3) in Table I. In the table, the particle hopping rates for noninteracting asymmetric random walkers are also shown. The process we have defined is different from the noninteracting particles in two respects. First, the rates for individual hopping are smaller than those for the noninteracting particles when there are two particles on a site ($20\rightarrow 11,02\rightarrow 11$). Second, two particles can hop simultaneously to the same nearest-neighboring site ($20\rightarrow 02,02\rightarrow 20$). To emphasize the latter property, we call the model the multiparticle-hopping asymmetric diffusion model (MADM).

The choice of the condition (7) is essential for the following computations. It will be shown that the Bethe equations of the MADM and the ASEP can be analyzed in a parallel fashion. Here, for later use, we introduce the so-called q number:

$$[n]=\frac{1-\gamma^n}{1-\gamma}. \tag{9}$$

Of course $[2]=1+\gamma$ in Eq. (8) is consistent with the definition (9). In the limit $\gamma\rightarrow 1$, $[n]$ simply reduces to n.

The master equation for the general N-particle case is defined similarly. The physical region of the coordinates $x_1,\ldots,x_N$ is $x_{j+1}-x_j\geqslant 1$ and $x_{j+1}\geqslant x_j$ for $j=1,\ldots,N-1$ for the ASEP and the MADM, respectively. The master equation is given by

$$\frac{d}{dt}P_N(x_1,\ldots,x_j,\ldots,x_N;t)=\sum_{j=1}^{N}[P_N(\ldots,x_j-1,\ldots;t)+\gamma P_N(\ldots,x_j+1,\ldots;t)-(1+\gamma)P_N(\ldots,x_j,\ldots;t)], \tag{10}$$

while the condition at the boundary of the physical region is

$$(1+\gamma)P_N(\ldots,x_j,x_j+1,\ldots;t)=P_N(\ldots,x_j,x_j,\ldots;t)+\gamma P_N(\ldots,x_j+1,x_j+1,\ldots;t)\quad (j=1,\ldots,N-1) \tag{11}$$

for the ASEP and

$$(1+\gamma)P_N(\ldots,x_j,x_j-1,\ldots;t)=P_N(\ldots,x_j-1,x_j-1,\ldots;t)+\gamma P_N(\ldots,x_j,x_j,\ldots;t)\quad (j=1,\ldots,N-1) \tag{12}$$

for the MADM.

For the MADM, one may wonder whether we can write down the master equation for $P_N(x_1,\ldots,x_N;t)$ when some x_j's are equal, only in terms of $P_N(x_1,\ldots,x_N;t)$'s in the physical region. It is possible to do so by repeated use of Eq. (12). In the case in which the consecutive M $(M\leqslant N)$ x_j's are equal, it is sufficient to know the equation for $P_M(x,\ldots,x;t)$ $(x_1=\cdots=x_M=x)$. For instance, the master equation for $P_N(x,x,x_3,\ldots,x_N;t)$ with $x_{j+1}>x_j (j=3,\ldots,N-1)$ and $x_3>x$ reads [cf. Eq. (8)]

$$\begin{aligned}\frac{d}{dt}P_N(x,x,x_3,\ldots,x_N;t)={}&P_N(x-1,x,x_3,\ldots,x_N;t)+\frac{1}{[2]}P_N(x-1,x-1,x_3,\ldots,x_N;t)+\gamma P_N(x,x+1,x_3,\ldots,x_N;t)\\&+\frac{\gamma^2}{[2]}P_N(x+1,x+1,x_3,\ldots,x_N;t)+\sum_{j=3}^{N}[P_N(x,x,\ldots,x_j-1,\ldots;t)+\gamma P_N(x,x,\ldots,x_j\\&+1,\ldots;t)]-\left[(N-1)(1+\gamma)+\frac{1}{[2]}+\frac{\gamma^2}{[2]}\right]P_N(x,x,x_3,\ldots,x_N;t).\end{aligned} \tag{13}$$

It turns out that the master equation for $P_N(x,\ldots,x;t)$ $(x_1=\cdots=x_N=x)$ can be rewritten in a compact form in terms of the q number:

$$\begin{aligned}&\frac{d}{dt}P_N(x,\ldots,x;t)\\&=\sum_{l=1}^{N}\frac{1}{[l]}P_N(\overbrace{x-1,\ldots,x-1}^{l},\overbrace{x,\ldots,x}^{N-l};t)+\sum_{l=1}^{N}\frac{\gamma^l}{[l]}P_N(\overbrace{x,\ldots,x}^{N-l},\overbrace{x+1,\ldots,x+1}^{l};t)\\&\quad-\sum_{l=1}^{N}\frac{1+\gamma^l}{[l]}P_N(x,\ldots,x;t).\end{aligned} \tag{14}$$

It is clear that more particles than one can hop simultaneously to the same nearest-neighboring site. A proof of Eq. (14) is shown in Appendix A.

III. CONSTRUCTION OF EIGENFUNCTION BY BETHE ANSATZ METHOD

Let the word ''energy'' mean the eigenvalue for the eigenstate of the process. It is not necessarily a real number. The real part of the energy corresponds to the decay rate of the eigenstate. Since the MADM is a stochastic model, the real part of the energy is larger than or equal to zero. The state with energy zero corresponds to the stationary state of the system. If we assume the existence of a unique stationary state, the system goes to the stationary state irrespective of the initial condition when the time $t\to\infty$. The stationary state of the MADM on the periodic chain is rather trivial for each particle number N: every possible configuration with N particles has an equal weight.

In order to study the time-dependent properties of the system, we need information about the excited states. In particular, the long time behaviors of the system strongly depend on the first excited state. In this section, we construct the eigenfunctions of the MADM by the Bethe ansatz method. The discussion below is analogous to those for the Heisenberg spin chain [23] and the ASEP [24]. The resulting Bethe equation will be analyzed in the next section.

We proceed with the $N=1$ and $N=2$ cases, and present the formulas for general N. First we consider the $N=1$ case. We substitute

$$P_1(x;t)=e^{-\epsilon_z t}z^x \tag{15}$$

into Eq. (2). The energy ϵ_z is easily calculated as

$$\epsilon_z=(1-z^{-1})+\gamma(1-z). \tag{16}$$

Because of the periodic boundary condition $P_1(x+L;t)=P_1(x;t)$, z should satisfy $z^L=1$.

Second, we consider the $N=2$ case. Assuming the time dependence as $\exp(-E_2t)$, we set in Eq. (4)

$$P_2(x_1,x_2;t)=e^{-E_2t}P_2(x_1,x_2), \tag{17}$$

where

$$P_2(x_1,x_2)=A_{12}z_1^{x_1}z_2^{x_2}+A_{21}z_2^{x_1}z_1^{x_2}. \tag{18}$$

The energy E_2 is the sum of the energies of the two particles,

$$E_2=\epsilon_{z_1}+\epsilon_{z_2}. \tag{19}$$

The condition (7) fixes the two-particle S-matrix A_{12}/A_{21} as

$$\frac{A_{12}}{A_{21}}=-\frac{1+\gamma z_1z_2-(1+\gamma)z_2}{1+\gamma z_1z_2-(1+\gamma)z_1}. \tag{20}$$

In the sequel, we employ the normalization

$$A_{jk}=\frac{1+\gamma z_jz_k-(1+\gamma)z_k}{z_j-z_k}. \tag{21}$$

The periodic boundary condition gives the Bethe equations for $N=2$,

$$z_1^L=-A_{12}/A_{21},\quad z_2^L=-A_{21}/A_{12}. \tag{22}$$

For the general N-particle case, we set in Eq. (10)

$$P_N(x_1,\dots,x_N;t)=e^{-E_Nt}P_N(x_1,\dots,x_N), \tag{23}$$

where

$$P_N(x_1,\dots,x_N)=\sum_{\sigma\in\mathfrak{S}_N}A_{\sigma(1)\cdots\sigma(N)}z_{\sigma(1)}^{x_1}\cdots z_{\sigma(N)}^{x_N}. \tag{24}$$

Here the symbol $\mathfrak{S}_N$ denotes all permutations of N numbers $\{1,\dots,N\}$ and σ is an element of $\mathfrak{S}_N$. The energy E_N is simply the sum of the energies of N particles,

$$E_N=\sum_{j=1}^{N}\epsilon_{z_j}. \tag{25}$$

The coefficients $A_{j_1,\dots,j_N}$ are fixed by Eqs. (12). They are written in a product of the coefficients A_{jk} (21),

$$A_{j_1,\dots,j_N}=\prod_{1\le k<l\le N}A_{j_kj_l}. \tag{26}$$

Imposing the periodic boundary condition on Eq. (24) gives the Bethe equation,

$$z_j^L=(-1)^{N-1}\prod_{l=1}^{N}\frac{1+\gamma z_jz_l-(1+\gamma)z_l}{1+\gamma z_jz_l-(1+\gamma)z_j} \tag{27}$$

for $j=1,\dots,N$. In the following sections, we shall solve the Bethe equation (27) in the thermodynamic limit and calculate the finite size correction for the gap of the energy (25).

Taking the logarithm of Eq. (27) and introducing a new variable α by

$$z^{-1}=\frac{1+\gamma e^{i\alpha}}{1+e^{i\alpha}}, \tag{28}$$

we have

$$p^0(\alpha_j)=\frac{2\pi}{L}I_j+\frac{1}{L}\sum_{l=1}^{N}\Theta(\alpha_j,\alpha_l). \tag{29}$$

Here we have defined the functions $p^0(\alpha)$ and $\Theta(\alpha,\beta)$ by

$$p^0(\alpha)=-i\ln\left[\frac{1+\gamma e^{i\alpha}}{1+e^{i\alpha}}\right], \tag{30}$$

$$\Theta(\alpha,\beta)=\Theta(\alpha-\beta)=-i\ln\left[\frac{\sinh\left[\nu+\frac{i}{2}(\alpha-\beta)\right]}{\sinh\left[\nu-\frac{i}{2}(\alpha-\beta)\right]}\right], \tag{31}$$

with $\gamma=\exp(-2\nu)$ $(0\le\nu<\infty)$. A choice of the set $\{I_j\}$ will be shown shortly. The advantage of using the variable α

is clear. To solve the integral equation resulting from the Bethe equation in the thermodynamic limit, it is essential that the function $\Theta(\alpha,\beta)$ depends on the difference of the arguments α and β. The function $\Theta(\alpha)$ appears also in the Bethe equation for the ASEP [16,17]. The only difference in the Bethe equations for the MADM and the ASEP is the explicit forms of the function $p^0(\alpha)$.

Different sets $\{I_j\}$ correspond to different energy eigenstates of the system. In particular, the stationary state corresponds to the choice

$$I_j = -\frac{N+1}{2} + j \tag{32}$$

for $j=1,\ldots,N$, whereas the first excited state is associated with the set

$$I_j = -\frac{N+1}{2} + j \tag{33}$$

for $j=1,\ldots,N-1$ and $I_N=(N+1)/2$.

IV. BETHE EQUATION IN THE THERMODYNAMIC LIMIT

In this section, we analyze the Bethe equation (29). The discussions proceed parallel to those for the asymmetric six-vertex model [16,17] in spite of the difference of the function $p^0(\alpha)$. As is often the case, the Bethe equation (29) cannot be solved explicitly for finite L and N. We consider the thermodynamic limit $L,N\to\infty$ with $N/L=\rho$ fixed. One then assumes that the solutions $\{\alpha_j\}$ for the Bethe equation are distributed densely along a smooth curve C in the complex α plane in the thermodynamic limit [10,11]. From the symmetry of the Bethe equation, the curve C is symmetric with respect to the imaginary axis. The endpoints of the curve C are denoted by $(-a+ib)$ and $(a+ib)$. Next, let $R(\alpha)L/2\pi$ denote the density of the roots on the curve. In addition, one defines a function $F(\alpha)$ such that $dF/d\alpha = R(\alpha)/2\pi$ along the curve with $F=0$ at the midpoint of C.

The normalization of the function $R(\alpha)$ is

$$\frac{1}{2\pi}\int_{-a+ib}^{a+ib} R(\alpha)\,d\alpha = \rho = F(a+ib) - F(-a+ib). \tag{34}$$

Since $F(-a+ib) = -F(a+ib)$ due to the above-mentioned symmetry, we have

$$F(a+ib) = \frac{\rho}{2}. \tag{35}$$

In the thermodynamic limit, the Bethe equation (29) has the form

$$p^0(\alpha) = 2\pi F(\alpha) + \frac{1}{2\pi}\int_{-a+ib}^{a+ib} \Theta(\alpha-\beta)R(\beta)\,d\beta. \tag{36}$$

Taking the derivative with respect to α in Eq. (36), we get the integral equation for the density of the roots, $R(\alpha)$:

$$R(\alpha) + \frac{1}{2\pi}\int_{-a+ib}^{a+ib} K(\alpha-\beta)R(\beta)\,d\beta = \zeta(\alpha), \tag{37}$$

where

$$\zeta(\alpha) = \frac{d}{d\alpha}p^0(\alpha) = -\frac{\sinh\nu}{\cosh\nu + \cos(\alpha+i\nu)}, \tag{38}$$

$$K(\alpha) = \frac{d}{d\alpha}\Theta(\alpha) = \frac{\sinh(2\nu)}{\cosh(2\nu)+\cos\alpha}. \tag{39}$$

We want to solve Eq. (37) and obtain the explicit expression of $R(\alpha)$.

We introduce the transformation $\alpha\to u = \alpha - ib$ $(\beta\to v = \beta - ib)$ to obtain integrals running over the real axis from $v=-a$ to $v=a$. After this transformation, the functions $p^0(\alpha)$ and $R(\alpha)$ depend on the parameter b and will be denoted by $p^0(u,b)$ and $R(u,b)$, respectively. Equations (36) and (37) are rewritten as

$$p^0(u,b) = 2\pi F(u,b) + \frac{1}{2\pi}\int_{-a}^{a}\Theta(u-v)R(v,b)\,dv, \tag{40}$$

$$R(u,b) + \frac{1}{2\pi}\int_{-a}^{a}K(u-v)R(v,b)\,dv = \zeta(u,b), \tag{41}$$

where

$$\zeta(u,b) = -\frac{\sinh\nu}{\cosh\nu + \cos[u+i(b+\nu)]}. \tag{42}$$

In the following we set $b>0$ and $a=-\pi$. The choices are valid for the ASEP and are expected to hold for the MADM as well from the numerical calculations for small L and N. We first solve the integral equation (41) by the Fourier transformation. We expand the functions $K(u),\zeta(u,b),R(u,b)$ in the Fourier series:

$$X(u) = \sum_{n=-\infty}^{\infty}\hat{X}_n e^{-inu}, \tag{43}$$

where X stands for K,ζ,R. The Fourier coefficients $\hat{K}_n$ are the same as those given in [16],

$$\hat{K}_n = e^{-2|n|\nu}, \tag{44}$$

and $\hat{\zeta}_n$ are calculated as

$$\hat{\zeta}_n = \begin{cases} 0 & (n\geq 0), \\ 2(-1)^{n-1}e^{(b+\nu)n}\sinh(n\nu) & (n<0). \end{cases} \tag{45}$$

In terms of the Fourier coefficients, the integral equation (41) is rewritten as

$$\hat{R}_n[1-\hat{K}_n] = \hat{\zeta}_n\,. \tag{46}$$

Hence, for $n\neq 0$, we get

$$\hat{R}_n = \begin{cases} 0 & (n>0) \\ (-)^n e^{bn} & (n<0). \end{cases} \tag{47}$$

Inserting Eq. (47) into Eq. (43), we find

$$R(u,b) = \hat{R}_0 - \frac{e^{iu-b}}{1+e^{iu-b}}. \tag{48}$$

The coefficient $\hat{R}_0$ is not yet determined since the Eq. (46) becomes trivial when $n=0$. The coefficient $\hat{R}_0$ is fixed by the condition $\hat{R}(\pi,b)=0$ [17] as

$$\hat{R}_0 = -\frac{e^{-b}}{1-e^{-b}}. \tag{49}$$

Next we express $\rho=N/L$ in terms of the parameter b. Setting $u=-\pi$ in Eq. (40) and using Eq. (35), we find

$$p^0(-\pi,b) = \pi\rho - \frac{1}{2\pi}\int_{-\pi}^{\pi}\Theta(-\pi-v)R(v,b)dv. \tag{50}$$

From (30), the left-hand side is shown to be

$$p^0(-\pi,b) = -i\ln\left[\frac{1-e^{-(b+2\nu)}}{1-e^{-b}}\right]. \tag{51}$$

The second term on the right-hand side is rewritten as

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\Theta(-\pi-v)R(v,b)dv = -\frac{1}{2\pi}\sum_{n=-\infty}^{\infty}\hat{R}_n J_n, \tag{52}$$

where J_n's were calculated in [17],

$$J_n = -\int_{-\pi}^{\pi}\Theta(-\pi-v)e^{-inv}dv$$
$$= \begin{cases} (-1)^n \dfrac{2\pi i}{n}(1-e^{-2n\nu}) & (n>0) \\ 2\pi^2 & (n=0) \\ (-1)^n \dfrac{2\pi i}{n}(1-e^{2n\nu}) & (n<0). \end{cases} \tag{53}$$

Using the explicit forms of the Fourier coefficients $\hat{R}_n$ (47), we have

$$-\frac{1}{2\pi}\sum_{n=-\infty}^{\infty}\hat{R}_n J_n = -\pi\hat{R}_0 + i[-\ln(1-e^{-b}) + \ln(1-e^{-(b+2\nu)})]. \tag{54}$$

Hence we obtain

$$\rho = -\hat{R}_0 = \frac{e^{-b}}{1-e^{-b}}. \tag{55}$$

We shall use the following expression for the function $R(u,b)$:

$$R(u,b) = -\rho - \frac{e^{-b+iu}}{1-e^{-b+iu}}. \tag{56}$$

So far, we have employed the variable α (or u) to solve the Bethe equation in the thermodynamic limit. To calculate the finite size correction of the energy gap, it is convenient to introduce another variable ξ by $\xi=\exp(i\alpha)=\exp(iu-b)$. In this variable, the Bethe equation (27) and the energy E_N (25) take the forms

$$\left[\frac{1+\gamma\xi_j}{1+\xi_j}\right]^L = (-)^{N-1}\prod_{l=1}^{N}\frac{\xi_j-\gamma\xi_l}{\xi_l-\gamma\xi_j}, \tag{57}$$

$$\frac{E_N}{1-\gamma} = \sum_{j=1}^{N}\left\{\frac{\xi_j}{1+\xi_j} - \frac{\gamma\xi_j}{1+\gamma\xi_j}\right\}, \tag{58}$$

respectively. Following [7], we introduce a function $Z_L(\xi)$ of the complex variable,

$$iZ_L(\xi) = \ln\left[\frac{1+\gamma\xi}{\xi^\rho(1+\xi)}\right] + \frac{1}{L}\sum_{l=1}^{N}\left\{\ln\xi_l - \ln\left[\frac{1-\gamma\xi_l/\xi}{1-\gamma\xi/\xi_l}\right]\right\}, \tag{59}$$

and its derivative,

$$Q_L(\xi) = i\xi\frac{d}{d\xi}Z_L(\xi). \tag{60}$$

We refer to $Z_L(\xi)$ as the phase function. Taking the logarithm of the Bethe equation (57) gives

$$Z_L(\xi_j) = \frac{2\pi}{L}I_j. \tag{61}$$

In the thermodynamic limit, the function $Q_N(\xi)$ is nothing but the function $R(u,b)$ (56) in terms of the variable ξ:

$$Q_\infty(\xi) = -\rho - \frac{\xi}{1+\xi}. \tag{62}$$

We can obtain the explicit expression of the phase function $Z_\infty(\xi)$ by integrating $Q_\infty(\xi)$ [cf. Eq. (60)]. The integration constant can be fixed as follows. In terms of the variable ξ, the solutions of the Bethe equations for the stationary state and its neighboring states form a closed contour; it starts from $e^{\pi i}\rho/(1+\rho)$, enclosing the origin clockwise and comes back to $e^{-\pi i}\rho/(1+\rho)$. If we define $\xi_c^0=e^{-\pi i}\rho/(1+\rho)$, we find

$$Z_\infty(\xi_c^0) = \pi\rho, \quad Z_\infty(e^{2\pi i}\xi_c^0) = -\pi\rho. \tag{63}$$

These relations can be used to fix the integration constant,

$$iZ_\infty(\xi) = -\ln[\xi^\rho(1+\xi)] + \ln\left[\frac{\rho^\rho}{(1+\rho)^{1+\rho}}\right]. \tag{64}$$

V. FINITE SIZE CORRECTION OF THE ENERGY GAP

In this section, we analyze the finite size correction of the energy (58) for the first excited state characterized by Eq. (33). Assuming appropriate properties of the phase function $Z_L(\xi)$ for general L, we expand the energy gap in powers of $L^{-1/2}$. The real part of the energy gap will be shown to start from the order $O(L^{-3/2})$. To obtain the finite size correction, we use the following formula:

$$\sum_{j=1}^{N} f(\xi_j) = -\frac{L}{2\pi i}\int f(\xi)Q_L(\xi)\frac{\mathrm{d}\xi}{\xi} + \frac{iy}{2}[f(\xi_c) - f(\xi_c e^{2\pi i})] + \sum_{m=1}^{\infty} A_m[f]Y_m(y)\varepsilon^m, \tag{65}$$

where the set $\{\xi_j\}$ is a solution of the Bethe equation and the expansion parameter ε is

$$\varepsilon = \sqrt{\frac{\pi}{L}}. \tag{66}$$

The derivation of the formula (65) is given in [7] and is briefly summarized in Appendix B with the definitions of $y, \xi_c, A_m[f]$, and $Y_m(y)$.

First we apply the formula for the phase function $Z_L(\xi)$ (59). Here we take $f = f_Z$, where

$$f_Z(\xi') = \ln\xi' - \ln\left[\frac{1-\gamma\xi'/\xi}{1-\gamma\xi/\xi'}\right] = \ln\xi' + \sum_{n\neq 0}\frac{1}{n}\gamma^{|n|}\left(\frac{\xi'}{\xi}\right)^n. \tag{67}$$

This then leads to an integral equation for $Q_L(\xi)$, which can be solved by the Fourier transformation method. Using Eqs. (65) and (67) in Eq. (59) gives

$$iZ_L(\xi) = \ln\left[\frac{1+\gamma\xi}{\xi^\rho(1+\xi)}\right] + \frac{1}{L}\sum_{j=1}^{N} f_Z(\xi_j) = -\rho\ln\xi + G(\xi) - \frac{1}{2\pi i}\int f_Z(\xi')Q_L(\xi')\frac{d\xi'}{\xi'}, \tag{68}$$

where

$$G(\xi) = \ln\left[\frac{1+\gamma\xi}{1+\xi}\right] + \delta - \frac{1}{\pi}\sum_{m=1}^{\infty}\sum_{k=1}^{m} b_{m,k}\left\{\sum_n \frac{(-1)^k(n+1)\cdots(n+k-1)}{k!\,\xi_c^k}\gamma^{|n|}\left(\frac{\xi}{\xi_c}\right)^n\right\}Y_m(y). \tag{69}$$

Here and hereafter Σ_n indicates the sum over all integers n. We obtain the expression for the phase function $Z_L(\xi)$,

$$iZ_L(\xi) = i\pi\rho + \delta - \frac{1}{\pi}\sum_{m=1}^{\infty}\sum_{k=1}^{m} b_{m,k}\frac{(-1)^k}{k\xi_c^k}Y_m(y)\varepsilon^{m+2} - \rho\ln\frac{\xi}{\xi_c} + \ln\frac{1+\xi_c}{1+\xi} + \frac{1}{\pi}\sum_{m=1}^{\infty}\sum_{k=1}^{m} b_{m,k}\{g_k(\xi) - g_k(\xi_c)\}Y_m(y)\varepsilon^{m+2}, \tag{70}$$

with

$$g_k(\xi) = \frac{(-1)^{k-1}}{k!\,\xi_c^k}\sum_{n\neq 0}(n+1)\cdots(n+k-1)\frac{\alpha^{|n|}}{1-\alpha^{|n|}}\left(\frac{\xi}{\xi_c}\right)^n. \tag{71}$$

If we set $\xi = \xi_c$ in Eq. (70), we find

$$0 = \sum_{m=1}^{\infty}\sum_{k=1}^{m} b_{m,k}\frac{(-1)^k}{k\xi_c^k}Y_m(y)\varepsilon^{m+2}, \tag{72}$$

due to Eq. (B1).

Having determined the phase function $Z_L(\xi)$, we next apply Eq. (65) to the energy (58). We set $f = f_E$, where $(1-\gamma)f_E(\xi)$ is the one-particle energy (16) in terms of the variable ξ. It reads

$$f_E(\xi) = \frac{\xi}{1+\xi} - \frac{\gamma\xi}{1+\gamma\xi} = \sum_{n=1}^{\infty}(-1)^{n-1}(1-\gamma^n)\xi^n. \tag{73}$$

Applying the summation formula (65) for the energy yields

$$\frac{E_N}{1-\gamma} = -\frac{L}{2\pi i}\int f_E(\xi)Q_L(\xi)\frac{d\xi}{\xi} + \frac{iy}{2}[f_E(\xi_c) - f_E(\xi_c e^{2\pi i})] + \sum_{m=1}^{\infty} A_m[f_E]Y_m(y)\varepsilon^m. \tag{74}$$

The first term on the right-hand side is calculated as

$$\frac{1}{2\pi i}\int f_E(\xi)Q_L(\xi)\frac{d\xi}{\xi} = \sum_{n=1}^{n} n(-1)^n G_{-n}, \tag{75}$$

where G_n denote the Fourier coefficients of $G(\xi)$:

$$G(\xi)=\sum_n G_n\xi^n. \tag{76}$$

Moreover, the definition of $A_m[f]$ (B7) together with the explicit form of $f_E(\xi)$ (73) gives

$$A_m[f_E]=-\sum_{k=1}^{m}\sum_{n=k}^{\infty} b_{m,k}\frac{n(n-1)\cdots(n-k+1)}{k!}\times(-1)^n(1-\gamma^n)\xi_c^{n-k}. \tag{77}$$

Thus we obtain

$$\frac{E_N}{1-\gamma}=\sum_{m=1}^{\infty}\sum_{k=1}^{m} b_{m,k}\left[-\frac{1}{1+\xi_c}\right]^{k+1}Y_m(y)\varepsilon^m =\sum_{m=2}^{\infty}\sum_{k=2}^{m} b_{m,k}C_k[\xi_c]Y_m(y)\varepsilon^m, \tag{78}$$

with

$$C_k[\xi_c]=(-1)^{k-1}\frac{(1+\xi_c)^{k-1}+k\xi_c^{k-1}}{k(1+\xi_c)^{k+1}\xi_c^{k-1}}. \tag{79}$$

Here we have used Eq. (72) to eliminate the $O(\varepsilon)$ term in the series.

For the calculation of the $O(\varepsilon^2)$ and $O(\varepsilon^3)$ terms, we can simply set $\xi_c=\xi_c^0=e^{-\pi i}\rho/(1+\rho)$. After some computations, we obtain

$$\frac{E_N}{1-\gamma}=2\pi i(1-2\rho)\frac{1}{L}+2(3+\rho)\sqrt{\frac{\rho}{1+\rho}}C\frac{1}{L^{3/2}}+\cdots, \tag{80}$$

where C is a constant given in [7]. Numerically, $C=6.509\ldots$. The expansion (80) shows that the real part of the energy gap is $O(L^{-3/2})$ in the lowest approximation. This suggests that the MADM belongs to the KPZ universality class.

VI. CONCLUDING REMARKS

In this paper, we have proposed an asymmetric diffusion model of many particles. The model is defined in terms of the master equation for the probability $P_N(x_1,\ldots,x_N;t)$ of finding N particles on lattice sites $x_1,\ldots,x_N$ at time t. The master equation is given by Eq. (10) with condition (12). Condition (12) is translated into the master equation for $P_N(x_1,\ldots,x_N;t)$ with some x_j's being equal. Especially the master equation for $P_N(x,\ldots,x;t)$ $(x_1=\cdots=x_N=x)$ turns out to be written in a compact form (14) using the q number (9). In contrast with the asymmetric simple exclusion process (ASEP), each lattice site can contain more than one particle. In addition, more particles than one can hop simultaneously to the left or right nearest neighboring site. This is the reason why we have called the model the multiparticle-hopping asymmetric diffusion model (MADM). The most remarkable feature of the model is that the model can be solved quite parallel to the ASEP. The eigenfunctions of the model are constructed by the conventional Bethe ansatz method. The resulting Bethe equation has been analyzed in the thermodynamic limit. The finite size correction of the energy gap is calculated to be $O(L^{-3/2})$, where L is the length of the chain. This scaling law is the same as that for the ASEP. It suggests that the model defined in this paper is in the Kardar-Parisi-Zhang universality class.

In this paper, we have only considered the model with stochastic interpretation. Similar analysis should be possible for a generalized model without such interpretation. It would be interesting to pursue the similarity, for instance, with the asymmetric XXZ chain. Besides, the Bethe equation for the q-boson model in [22] can be analyzed in a similar manner, though some modifications are needed. The problem is now under investigations and the results will be reported in future publications.

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APPENDIX A: MASTER EQUATION WITH SAME VARIABLES

In this appendix, we confirm Eq. (14) from Eqs. (10) and (12). First we prove the relations

$$\begin{aligned} P_N(x,\overbrace{x-1,\ldots,x-1}^{N-1};t)&=\frac{[N-1]}{[N]}P_N(x-1,\ldots,x-1;t)+\frac{\gamma^{N-1}}{[N]}P_N(x,\ldots,x;t),\\ P_N(\overbrace{x,\ldots,x}^{N-1},x-1;t)&=\frac{1}{[N]}P_N(x-1,\ldots,x-1;t)+\gamma\frac{[N-1]}{[N]}P_N(x,\ldots,x;t), \end{aligned} \tag{A1}$$

by the induction. For $N=2$ they reduce to a single defining relation (7). We assume that the relations are valid for N. From the relations for N, we have

$$P_{N+1}(x,\overbrace{x-1,\ldots,x-1}^{N};t)=\frac{[N-1]}{[N]}P_{N+1}(x-1,\ldots,x-1;t)+\frac{\gamma^{N-1}}{[N]}P_{N+1}(\overbrace{x,\ldots,x}^{N},x-1;t),$$
$$P_{N+1}(\overbrace{x,\ldots,x}^{N},x-1;t)=\frac{1}{[N]}P_{N+1}(x,\overbrace{x-1,\ldots,x-1}^{N};t)+\gamma\frac{[N-1]}{[N]}P_{N+1}(x,\ldots,x;t). \tag{A2}$$

Solving the system of the equations and using the formula $[N-1][N+1]+\gamma^{N-1}=[N]^2$ for the q number (9), we obtain the relations (A1) for $N+1$.

Now we prove Eq. (14) again by the induction. For $N=2$, Eq. (14) reduces to Eq. (8). We assume Eq. (14) is valid for N and derive it for $N+1$. We want to set $x_1=\cdots=x_{N+1}=x$ in Eq. (10) for the $N+1$ case. This is done in two steps. For the function $P_{N+1}(x_1,\cdots,x_{N+1};t)$ with $x_1=\ldots=x_N=x$, we can use Eq. (14) for N. We then find

$$\begin{aligned}
&\frac{d}{dt}P_{N+1}(x,\ldots,x,x_{N+1};t)\\
&=\sum_{l=1}^{N}\frac{1}{[l]}P_{N+1}(\overbrace{x-1,\ldots,x-1}^{l},\overbrace{x,\ldots,x}^{N-l},x_{N+1};t)\\
&\quad+\sum_{l=1}^{N}\frac{\gamma^l}{[l]}P_{N+1}(\overbrace{x,\ldots,x}^{N-l},\overbrace{x+1,\ldots,x+1}^{l},x_{N+1};t)\\
&\quad-\sum_{l=1}^{N}\frac{1+\gamma^l}{[l]}P_{N+1}(x,\ldots,x,x_{N+1};t)\\
&\quad+P_{N+1}(x,\ldots,x,x_{N+1}-1;t)\\
&\quad+\gamma P_{N+1}(x,\ldots,x,x_{N+1}+1;t)\\
&\quad-(1+\gamma)P_{N+1}(x,\ldots,x,x_{N+1};t).
\end{aligned}\tag{A3}$$

Setting $x_{N+1}=x$ in Eq. (A3), using the second equation of Eqs. (A1) and noticing the formula

$$\sum_{l=1}^{N}\frac{\gamma^l}{[l][l+1]}=1-\frac{1}{[N+1]},\tag{A4}$$

for the q number, we obtain Eq. (14) for $N+1$. Thus we have proved Eq. (14) for any $N(\geqslant 2)$.

APPENDIX B: SUMMATION FORMULA

In this appendix, we give some definitions related to the summation formula (65). First we assume that the phase function $Z_L(\xi)$ for finite L has a vanishing derivative at a certain value of ξ and define ξ_c by the relation $dZ_L(\xi_c)/d\xi=0$. Since $dZ_\infty(\xi_c^0)/d\xi=0$, ξ_c is expected to be close to ξ_c^0 for large L. In addition, define δ by the relation

$$Z_L(\xi_c)=\pi\rho-i\delta.\tag{B1}$$

Comparing Eq. (B1) with Eq. (63), we see that δ becomes small for large L. Assuming the analyticity of $dZ_L(\xi)/d\xi$ at ξ_c, the inverse function Z_L^{-1} of $Z_L(\xi)$ is put in the form

$$Z_L^{-1}(\pi\rho-\xi)=\xi_c+\sum_{m=1}^{\infty}a_m(-i\sqrt{\delta+i\xi})^m.\tag{B2}$$

Consider the sum of the form $\Sigma_{j=1}^N f(\xi_j)$ where the set $\{\xi_j\}$ is a solution of the Bethe equation (57). Due to Eq. (61), the sum is written in the form

$$\sum_{j=1}^{N}f(\xi_j)=\sum_{j=1}^{N}f\left(Z_L^{-1}\left(\frac{2\pi}{L}I_j\right)\right).\tag{B3}$$

To evaluate the sum, we use a formula [6]

$$\sum_{j=1}^{N}f(j)=\int_1^N f(t)dt+\frac{1}{2}[f(N)+f(1)]+2\int_0^\infty\frac{\tilde{f}(N,t)-\tilde{f}(1,t)}{e^{2\pi t}-1}dt,\tag{B4}$$

where $\tilde{f}(s,t)=[f(s+it)-f(s-it)]/2i$. After some calculations, we obtain Eq. (65) with the definitions of y, $Y_m(y)$, and $A_m[f]$,

$$y=\delta L/\pi,\tag{B5}$$

$$\begin{aligned}
Y_m(y)&=Y_m^0(y)+(-i\sqrt{y-i})^m-(-i\sqrt{y+i})^m,\\
Y_m^0(y)&=\mathrm{Re}\left(\frac{m+2iy}{m+2}(-i\sqrt{y+i})^m\right.\\
&\quad\left.+\frac{1}{i}\int_0^\infty\frac{(-i\sqrt{y+i+t})^m-(-i\sqrt{y+i-t})^m}{e^{\pi t}-1}dt\right),
\end{aligned}\tag{B6}$$

$$A_m[f]=\sum_{k=1}^{m}\frac{b_{m,k}}{k!}f^{(k)}(\xi_c),\tag{B7}$$

$$\left(\sum_{m=1}^{\infty}a_m\varepsilon^m\right)^k=\sum_{m=k}^{\infty}b_{m,k}\varepsilon^m,\tag{B8}$$

where $f^{(k)}$ denotes the kth derivative of $f(\xi)$.

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